ON AN EXTENSION OF THE H^k MEAN CURVATURE FLOW

YI LI

ABSTRACT. In this note we generalize an extension theorem in [5] and [9] of the mean curvature flow to the H^k mean curvature flow under some extra conditions. The main difficult problem in proving the extension theorem is to find a suitable version of Michael-Simon inequality for the H^k mean curvature flow, and to do a suitable Moser iteration process. These two problems are overcame by imposing some extra conditions which may be weakened or removed in our forthcoming paper [7]. On the other hand, we derive some estimates for the generalized mean curvature flow, which have their own interesting.

1. INTRODUCTION

Let M be a compact n-dimensional hypersurface without boundary, which is smoothly embedded into the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ by the map

$$F_0 : M \to \mathbb{R}^{n+1}. \tag{1.1}$$

The generalized mean curvature flow (GMCF), an evolution equation of the mean curvature $H(\cdot, t)$, is a smooth family of immersions $F(\cdot, t) : M \to \mathbb{R}^{n+1}$ given by

$$\frac{\partial}{\partial t} F(\cdot, t) = -f(H(\cdot, t))\nu(\cdot, t), \qquad F(\cdot, 0) = F_0(\cdot), \tag{1.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, depending only on $H(\cdot, t)$, with some properties to guarantee the short time existence, and $\nu(\cdot, t)$ is the outer unit normal on $M_t := F(M, t)$ at $F(\cdot, t)$. The short time existence of the GMCF has been established in [8]. Namely, if $f' > 0$ along the GMCF, then it always admits a smooth solution on a maximal time interval $[0, T_{\max})$ with $T_{\max} < \infty$. Setting f the identity function is the classical mean curvature flow; on the other hand, if we choose $f(x)$ to be some power function x^k, then one gets the H^k mean curvature flow. In this note we mainly focus on the H^k mean curvature flow, but partly results on the GMCF are also derived.

In general, Huisken [3] proved that the mean curvature flow develops to singularities in finite time: Suppose that $T_{\max} < \infty$ is the first singularity time for the mean curvature flow. Then $\sup_{M_t} |A|(t) \to \infty$ as $t \to T_{\max}$.

Recently, Cooper [1], Le-Sesum [5], and Xu-Ye-Zhao [9] proved an extension theorem on the mean curvature flow under some curvature conditions. A natural question is whether we can generalize it to the GMCF, in particular, the H^k mean curvature flow. In this note, we give a partial answer to this question.

Theorem 1.1. *Suppose that the integers n and k are greater than or equal to 2 and that $n+1 \geq k$. Suppose that M is a compact n-dimensional hypersurface without boundary, smoothly embedded into $\mathbb{R}^{n+1}$ by the function F_0. Consider the H^k mean*

curvature flow on M

$$\frac{\partial}{\partial t}F(\cdot,t) = -H^k(\cdot,t)\nu(\cdot,t), \quad F(\cdot,0) = F_0(\cdot).$$

If

(a) $h_{ij}(t) \geq C g_{ij}(t)$ *along the H^k mean curvature flow for an uniform constant $C > 0$,*
(b) *for some $\alpha \geq n+k+1$,*

$$\|H(t)\|_{L^\alpha(M\times[0,T_{\max}))} := \left(\int_0^{T_{\max}}\int_{M_t} |H(t)|^\alpha_{g(t)} d\mu(t)dt\right)^{\frac{1}{\alpha}} < \infty,$$

then the flow can be extended over the time $T_{\max}$.

Remark 1.2. *When $k = 1$, $n+1 \geq k$ is trivial and the condition (a) should be weaken to be $h_{ij}(t) \geq -Cg_{ij}(t)$ for some uniform constant $C > 0$ (see* [5] *and* [9]*). we don't know the condition $n+1 \geq k$ is necessary, but in this note it is a technique assumption when we use the similar method in* [5]*. In the forthcoming paper* [7]*, we want to at least weaken the condition (a) and to remove the assumption $n+1 \geq k$.*

For the generalized mean curvature flow, we have the following two interesting estimates.

Theorem 1.3. *Suppose that the integers n and k are greater than or equal to 2. Suppose that M is a compact n-dimensional hypersurface without boundary, smoothly embedded into $\mathbb{R}^{n+1}$ by the function F_0. Consider the GMCF*

$$\frac{\partial}{\partial t}F(\cdot,t) = -f(H(\cdot,t))\nu(\cdot,t), \quad F(\cdot,0) = F_0(\cdot), \quad 0 \leq t \leq T \leq T_{\max} < \infty.$$

Suppose that $f \in C^\infty(\Omega)$ for an open set $\Omega \subset \mathbb{R}$, and that v is a smooth function on $M \times [0,T]$ such that its image is contained in Ω. Consider the differential inequality

$$\left(\frac{\partial}{\partial t} - \Delta_{f,t}\right)v \leq G\cdot f(v) + f''(v)|\nabla_t v|^2_{g(t)}, \quad v \geq 0, \quad G \in L^q(M\times[0,T]). \tag{1.3}$$

Let

$$C_{0,q} = \|f'(v)G\|_{L^q(M\times[0,T])}, \quad C_1 = \left(1 + \|H\|^{n+k+1}_{L^{n+k+1}(M\times[0,T])}\right)^{\frac{1}{k}},$$

and also let

$$\gamma = 2 + \frac{(k+1)^2}{k^2 n}.$$

We denote by $\mathcal{S}$ the set of all functions $f \in C^\infty(\Omega)$, where $\Omega \subset \mathbb{R}$ is the domain of f, satisfying

(i) *f satisfies the differential inequality (1.3),*
(ii) *$f'(x) > 0$ for all $x \in \Omega$,*
(iii) *$f(x) \geq 0$ whenever $x \geq 0$,*
(iv) *$f(H(t))H(t) \geq 0$ along the GMCF.*
(v) *$f'(v) \geq C_2 > 0$ on $M \times [0,T]$ for some uniform constant C_2.*

For any $\beta \geq 2$ and $q > \frac{\gamma}{\gamma-2}$, there exists a positive constant $C_{n,k,T}(C_{0,q}, C_1, \beta, q)$, depending only on n, k, T, β, q, $C_{0,q}$, C_1, and $\mathrm{Vol}(M)$, such that, for any $f \in \mathcal{S}$,

$$\begin{aligned} & \|\eta^2 f^\beta(v)\|_{L^{\gamma/2}(M\times[0,T])} \\ \leq\ & C_{n,k,T}(C_{0,q}, C_1, \beta, q) \left\| f^\beta(v) \left[\eta^2 + 2\eta \left(\frac{\partial}{\partial t} - f'(v)\Delta_t\right)\eta \right.\right. \\ +\ & \left.\left. \left(\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)} + \frac{8\beta^2-2\beta+2}{\beta(\beta-1)} f'(v)\right) |\nabla_t \eta|^2_{g(t)}\right]\right\|_{L^1(M\times[0,T])} \end{aligned}$$

where (the definition of $B_{n,k,T}$ is given in Section 3)

$$C_{n,k,T}(C_{0,q}, C_1, \beta, q) = \frac{\beta}{\beta-1}\max\left\{2(\widetilde{B}_{n,k,T}C_1)^{2/\gamma}, \left(2C_{0,q}\frac{\beta^2}{\beta-1}(\widetilde{B}_{n,k,T}C_1)^{2/\gamma}\right)^{1+\nu}\right\},$$

$\nu = \frac{\gamma}{(\gamma-2)q-\gamma}$, and η is any smooth function on $M\times[0,T]$ with the property that $\eta(x,0) = 0$ for all $x \in M$. In particular, if $f'(v)G \in L^\infty(M\times[0,T])$, then, letting $q \to \infty$, we have

$$\begin{aligned} C_{n,k,T}(C_{0,\infty}, C_1, \beta, \infty) &= \frac{2\beta}{\beta-1}\max\left\{1, \frac{C_{0,\infty}\beta^2}{\beta-1}\right\}(\widetilde{B}_{n,k,T}C_1)^{2/\gamma} \\ &\leq \left[8\max\{1, C_{0,\infty}\}\widetilde{B}^{2/\gamma}_{n,k,T}\right]\beta C_1^{2/\gamma}, \end{aligned}$$

where

$$\widetilde{B}_{n,k,T} = B_{n,k,T}\cdot\max\left\{\left(\frac{1}{C_2}\right)^{\frac{k+1}{2k}}, 1\right\}, \quad C_{0,\infty} = \|f'(v)G\|_{L^\infty(M\times[0,T])},$$

since $\frac{\beta}{\beta-1} \leq 2$; in this case, we obtain

$$\begin{aligned} & \|\eta^2 f^\beta(v)\|_{L^{\gamma/2}(M\times[0,T])} \\ \leq\ & D_{n,k,T}\beta C_1^{2/\gamma} \left\| f^\beta(v) \left[\eta^2 + 2\eta \left(\frac{\partial}{\partial t} - f'(v)\Delta_t\right)\eta \right.\right. \\ +\ & \left.\left. \left(\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)} + \frac{8\beta^2-2\beta+2}{\beta(\beta-1)} f'(v)\right) |\nabla_t \eta|^2_{g(t)}\right]\right\|_{L^1(M\times[0,T])}, \end{aligned}$$

where $D_{n,k,T} = 8\max\{1, C_{0,\infty}\}\widetilde{B}^{2/\gamma}_{n,k,T}$.

Corollary 1.4. *Suppose that the integers n and k are greater than or equal to 2. Suppose that M is a compact n-dimensional hypersurface without boundary, smoothly embedded into $\mathbb{R}^{n+1}$ by the function F_0. Consider the GMCF*

$$\frac{\partial}{\partial t}F(\cdot,t) = -f(H(\cdot,t))\nu(\cdot,t), \quad F(\cdot,0) = F_0(\cdot), \quad 0 \leq t \leq T \leq T_{\max} < \infty.$$

Suppose that $f \in C^\infty(\Omega)$ for an open set $\Omega \subset \mathbb{R}$, and that v is a smooth function on $M\times[0,T]$ such that its image is contained in Ω. Consider the differential inequality

$$\left(\frac{\partial}{\partial t} - \Delta_{f,t}\right)v \leq G\cdot f(v) + f''(v)|\nabla_t v|^2_{g(t)}, \quad v \geq 0, \quad G \in L^q(M\times[0,T]). \tag{1.4}$$

Let

$$C_{0,\infty} = \|f'(v)G\|_{L^\infty(M\times[0,T])}, \quad C_1 = \left(1 + \|H\|^{n+k+1}_{L^{n+k+1}(M\times[0,T])}\right)^{\frac{1}{k}},$$

and also let

$$\gamma = 2 + \frac{(k+1)^2}{k^2 n}.$$

We denote by $\mathcal{S}$ the set of all functions $f \in C^\infty(\Omega)$, where $\Omega \subset \mathbb{R}$ is the domain of f, satisfying

(i) *f satisfies the differential inequality (1.4),*
(ii) *$f'(x) > 0$ for all $x \in \Omega$,*
(iii) *$f(x) \geq 0$ whenever $x \geq 0$,*
(iv) *$f(H(t))H(t) \geq 0$ along the GMCF.*
(v) *$f'(v) \geq C_2 > 0$ on $M \times [0,T]$ for some uniform constant C_2.*

There exists an uniform constant $C_n > 0$, depending only on n, such that for any $\beta \geq 2$ and $f \in \mathcal{S}$ we have

$$\|f(v)\|_{L^\infty\left(M\times\left[\frac{T}{2},T\right]\right)} \leq E_{n,k,T}(\beta) \cdot C_1^{\frac{1}{\beta}\frac{2}{\gamma-2}} \cdot \|f(v)\|_{L^\beta(M\times[0,T])},$$

where

$$E_{n,k,T}(\beta) = (D_{n,k,T}C_n\beta)^{\frac{1}{\beta}\frac{\gamma}{\gamma-2}} \cdot \left(\frac{\gamma}{2}\right)^{\frac{1}{\beta}\frac{2\gamma}{(\gamma-2)^2}} \cdot 4^{\frac{1}{\beta}\frac{\gamma^2}{(\gamma-2)^2}},$$

and the constant $D_{n,k,T}$ is given in theorem 1.3.

Convention. If $f(x) : \mathbb{R} \to \mathbb{R}$ is a smooth function, $v(t)$ is another smooth function, throughout this note we denote by $f'(v)$ the value of $f'(x)$ at $x = v(t)$, namely,

$$f'(v) := \frac{d}{dx} f(x)\big|_{x=v}.$$

When we write $\frac{d}{dt}f(v)$, it means that

$$\frac{d}{dt}f(v(t)) = \frac{d}{dx}f(x)\Big|_{x=v(t)} \cdot \frac{d}{dt}v(t) = f'(v(t))v'(t).$$

For example, if $f(x) = x^k$, then

$$f'(v) = kv^{k-1}, \quad \frac{d}{dt}f(v) = kv^{k-1}v'.$$

Acknowledgements. The paper was written by author discussing the Fourier analysis and number theory with Andrew Ostergaard who asked him this problem. The author would particularly like to thank Daniel Paradis for many discussions on complex analysis and differential geometry during the summer.

2. Evolution equations for GMCF

In this section we fix our notation and derive some evolution equations for the GMCF. Let $g = \{g_{ij}\}$ be the induced metric on M obtained by pullbacking the standard metric $g_{\mathbb{R}^{n+1}}$ of $\mathbb{R}^{n+1}$. We denote by $A = \{h_{ij}\}$ the second fundamental form and $d\mu = \sqrt{\det(g_{ij})}$ the volume form on M, respectively. Using the local coordinates system and above notation, the mean curvature can be expressed as

$$H = g^{ij}h_{ij}. \tag{2.1}$$

For any two mixed tensors, say $T = \{T^i_{jk}\}$ and $S = \{S^i_{jk}\}$, their inner product relative to the induced metric g is given by

$$\langle T^i_{jk}, S^i_{jk}\rangle_g = g_{is}g^{jr}g^{ku}T^i_{jk}S^s_{ru}. \tag{2.2}$$

Then the norm of the tensor T is written as

$$|T|_g^2 = \langle T^i_{jk}, T^i_{jk}\rangle_g. \tag{2.3}$$

Using this notion, we have $|A|_g^2 = g^{ij}g^{kl}h_{ik}h_{jl}$. If $x_1, \cdots, x_n$ are local coordinates on M, one has

$$g_{ij} = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j}\right\rangle_{\mathbb{R}^{n+1}}, \qquad h_{ij} = -\left\langle \nu, \frac{\partial^2 F}{\partial x_i \partial x_j}\right\rangle_{\mathbb{R}^{n+1}}, \tag{2.4}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^{n+1}}$ denotes the Euclidean inner product of $\mathbb{R}^{n+1}$. Let ∇ denote the induced Levi-Civita connection on M. Hence for an vector $X = \{X^i\}$ we have

$$\nabla_j X^i = \frac{\partial}{\partial x_j}X^i + \Gamma^i_{jk}X^k, \tag{2.5}$$

where Γ^i_{jk} is the Christoffel symbol locally given by

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\right). \tag{2.6}$$

The induced Laplacian operator Δ on M is defined by

$$\Delta T^i_{jk} := g^{mn}\nabla_m\nabla_n T^i_{jk}. \tag{2.7}$$

Moreover, the Laplacian operator Δh_{ij} can be written as

$$\Delta h_{ij} = \nabla_i\nabla_j H + Hh_{il}g^{lm}h_{mj} - |A|_g^2 h_{ij}. \tag{2.8}$$

We write $g(t) = \{g_{ij}(t)\}, A(t) = \{h_{ij}(t)\}, \nu(t), H(t), d\mu(t), \nabla_t$, and Δ_t the corresponding induced metric, second fundamental form, outer unit normal vector, mean curvature, volume form, induced Levi-Civita connection, and induced Laplacian operator at time t. The position coordinates are not explicitly written in the above symbols if there is no confusion.

Proposition 2.1. (Evolution equations) *For the GMCF, one has*

$$\begin{aligned}
\frac{\partial}{\partial t}F(t) &= -f(H(t))\nu(t),\\
\frac{\partial}{\partial t}g_{ij}(t) &= \nabla_t f(H(t)) = f'(H(t))\cdot\nabla_t H(t),\\
\frac{\partial}{\partial t}h_{ij}(t) &= f'(H(t))\cdot\Delta_t h_{ij}(t) + f''(H(t))\nabla_i H\cdot\nabla_j H(t)\\
&\quad - [f(H(t)) + f'(H(t))H(t)]h_{il}(t)g^{lm}(t)h_{mj}(t) + f'(H(t))|A(t)|^2_{g(t)}h_{ij}(t),\\
\frac{\partial}{\partial t}H(t) &= f'(H(t))\Delta_t H(t) + f(H(t))|A(t)|^2_{g(t)} + f''(H(t))|\nabla_t H(t)|^2_{g(t)},\\
\frac{\partial}{\partial t}d\mu(t) &= -f(H(t))H(t)d\mu(t).
\end{aligned}$$

Proof. The proof is straightforward, but is more tedious than that in the classical setting. □

From the evolution equation for the mean curvature $H(t)$, it is natural to introduce the generalized Laplacian operator associated to the function f. Put

$$\Delta_{f,t}(\cdot) := f'(\cdot)\Delta_t(\cdot). \tag{2.9}$$

Hence

$$\frac{\partial}{\partial t} H(t) = \Delta_{f,t} H(t) + f(H(t))|A(t)|^2_{g(t)} + f''(H(t))|\nabla_t H(t)|^2_{g(t)}. \tag{2.10}$$

It is a special case of the following differential inequality

$$\left(\frac{\partial}{\partial t} - \Delta_{f,t}\right) v \leq G \cdot f(v) + f''(v)|\nabla_t v|^2_{g(t)}, \tag{2.11}$$

which is also discussed in [4].

3. A version of Michael-Simon inequality

Let us consider that M is the standard sphere S^n which is immersed into $\mathbb{R}^{n+1}$ by F_0. Just as in Example 2.1 [5], the H^k mean curvature flow with initial data F_0 has the formula $F(t) = r(t)F_0$. Hence

$$\frac{dr(t)}{dt} = -\frac{n^k}{r^k(t)}, \quad r(0) = 1.$$

This ODE gives $r(t) = [1 - (k+1)n^k t]^{\frac{1}{k+1}}$. The maximal time is $T_{\max} = \frac{1}{(k+1)n^k}$. Using $T_{\max}$ we can rewrite $r(t)$ as

$$r(t) = [(k+1)n^k(T_{\max} - t)]^{\frac{1}{k+1}}.$$

Hence the L^α-norm of $H(t)$ on $M \times [0, T]$ is

$$\|H(t)\|^\alpha_{L^\alpha(M\times[0,T_{\max}))} = \frac{n^\alpha \omega_n}{[(k+1)n^k]^{\frac{\alpha-n}{k+1}}} \int_0^{T_{\max}} \frac{dt}{(T-t)^{\frac{\alpha-n}{k+1}}},$$

which is finite if $\alpha < n+k+1$. Here ω_n denotes the area of S^n. It implies that the constant α in Theorem 1.1 is optional. When $\alpha = n+k+1$, we consider a rescaling transformation

$$\widetilde{F}(\cdot, t) = Q^\beta F\left(\cdot, \frac{t}{Q^\gamma}\right).$$

In order to make sure that $\|H(t)\|_{L^{n+k+1}(M\times[0,T_{\max}))}$ is invariant under this transformation, we must have

$$\gamma = \beta(k+1).$$

In particular, $\|H(t)\|_{L^{n+k+1}(M\times[0,T_{\max}))}$ is invariant under the following rescaling transformation

$$\widetilde{F}(\cdot, t) = Q \cdot F\left(\cdot, \frac{t}{Q^{k+1}}\right). \tag{3.1}$$

Remark 3.1. *In general, we consider the rescaling transformation of the GMCF*

$$\widetilde{F}(\cdot, t) = Q^\beta F\left(\cdot, \frac{t}{Q^\gamma}\right).$$

In order to guarantee that the quantity $\|H(t)\|_{L^\alpha(M\times[0,T_{\max}))}$ is invariant under this rescaling, we must have, for any x and $Q > 0$,

$$\gamma = (\alpha - n)\beta, \quad f(x) = Q^{\gamma-\beta} f\left(\frac{x}{Q^\beta}\right).$$

Letting $k = \alpha - n - 1$, we obtain

$$f(x) = Q^{k\beta} f\left(\frac{x}{Q^\beta}\right), \quad x \in \mathbb{R}, \quad Q > 0. \tag{3.2}$$

A solution for this functional equation is $f(x) = x^k$. Actually, we can show that the functional equation (3.2) has the unique solution with the form $f(x) = f(1)x^k$. Indeed[1]*, if we let $y = 1/Q$, then*

$$y^{k\beta} f(x) = f(xy^{\beta});$$

putting $x = 1$ gives $f(y^{\beta}) = f(1)y^{\beta}$ and hence $f(x) = f(1)x^k$. This is a reason why we restrict ourself to the H^k mean curvature flow.

The key step in [5] is to establish a version of Michael-Simon inequality. When $k = 1$, this type of equality has been proved in [5]. Considering the H^k mean curvature flow, one should generalize the Michael-Simon inequality to a "nonlinear" version when $k \geq 2$. The first trying step is how to find a suitable "nonlinear" number Q satisfying the property that it reduces to the original definition (that is, $Q = \frac{n}{n-2}$) when k equals 1. There are lots of such choices on this step, for instance, $Q = \frac{n}{n-k-1}, \frac{kn}{kn-2}, \frac{kn}{kn-(k+1)}$, etc. The first two numbers are easily to think about, but the third one is not so easily to find out, since there are at least two rules to obey: one should be compatible with the Hölder's inequality, Young's inequality, and interpolation inequality in the process of the proof; the second one is that we should find an analogous inequality which is the original one when $k = 1$.

Remark 3.2. *Here we give a heuristical proof why we chose $Q = \frac{kn}{kn-(k+1)}$. Starting from $w = v^a$ with some constant a determined later and using the original Michael-Simon inequality (see below) we have (in the following estimates we omit constants in each step)*

$$\left(\int_M v^{\frac{\alpha n}{n-1}} d\mu\right)^{\frac{n-1}{n}} \leq \int_M \left(|\nabla v| v^{a-1} + |H| v^a\right) d\mu.$$

From Hölder's inequality and Young's inequality, one has

$$\begin{aligned}
\left(\int_M v^{\frac{an}{n-1}} d\mu\right)^{\frac{n-1}{an}\frac{1}{b}} &\leq \left(\int_M (|\nabla v| v^{a-1} + |H| v^a) d\mu\right)^{\frac{1}{ab}}, \\
&\leq \|\nabla v\|^{\frac{1}{ab}}_{L^p(M)} \|v\|^{\frac{a-1}{ab}}_{L^{(a-1)q}(M)} + \|H\|^{\frac{1}{ab}}_{L^r(M)} \|v\|^{\frac{1}{b}}_{L^{as}(M)} \\
&\leq \|v\|^{\frac{(a-1)\alpha}{ab}}_{L^{(a-1)q}(M)} + \|\nabla v\|^{\frac{\beta}{ab}}_{L^p(M)} + \|H\|^{\frac{1}{ab}}_{L^r(M)} \|v\|^{\frac{1}{b}}_{L^{as}(M)},
\end{aligned}$$

where we put the wight $\frac{1}{b}$ on both sides (the reason will be seen soon), and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = \frac{1}{\alpha} + \frac{1}{\beta} = 1, \qquad p, q, r, s, \alpha, \beta > 1.$$

We let

$$\frac{1}{b} = \frac{(a-1)\alpha}{ab}, \qquad \frac{an}{n-1} = (a-1)q.$$

Therefore. $a = \frac{q(n-1)}{q(n-1)-n}$ and $\alpha = \frac{q(n-1)}{n}$. Moreover

$$\frac{an}{n-1} = \frac{qn}{(q-1)n - q}.$$

If $q = k + 1$, then we get

$$\frac{an}{n-1} = \frac{(k+1)n}{kn-(k+1)} = \frac{k+1}{k} \cdot \frac{kn}{kn-(k+1)}.$$

[1] Andrew told me this short proof.

There are two reasons to set $\frac{1}{b} = \frac{k+1}{k}$*: the first one comes from the careful investigation of the term* $\|H\|^{1/ab}_{L^r(M)}\|v\|^{1/b}_{L^{as}(M)}$ *by using the interpolation inequality, and the another reason is the equation* $\frac{1}{c} + \frac{kn-(k+1)}{kn} = 1$ *which gives* $c = \frac{kn}{k+1}$*. However, other reasons, e.g.,* $\frac{1}{p} + \frac{1}{k+1} = 1$ *determining* $p = \frac{k+1}{k}$*, can be seen in the detailed analysis of the proof. The above is an exploration for finding a suitable number* Q*, and, of course, is very naive and rough.*

Let M be a compact n-dimensional hypersurface without boundary, which is smoothly embedded in $\mathbb{R}^{n+1}$. The original Michael-Simon inequality states that for any nonnegative, C^1-functions w, one has

$$\left(\int_M w^{\frac{n}{n-1}}d\mu\right)^{\frac{n-1}{n}} \leq c_n \int_M (|\nabla w| + |H|w)d\mu. \tag{3.3}$$

Here c_n is the constant depending only on n. More precisely,

$$c_n = \frac{4^{n+1}}{\omega_n^{1/n}}, \qquad \omega_n = \mathrm{Area}(S^n). \tag{3.4}$$

Before proving the main theorem in this section, we state some elementary integral inequalities which can be proven by Hölder's inequality.

Lemma 3.3. *For any compact manifold* M *and any Lipschitz functions* f*, one has*

(i) $\|f\|_{L^p(M)} \leq \|f\|_{L^q(M)} \cdot \mathrm{Vol}(M)^{\frac{q-p}{pq}}$ *whenever* $0 < p < q$.

(ii) *for any* $k \geq 1$*, one has*

$$\int_M |f|^{1/k}d\mu \leq \left(\int_M |f|d\mu\right)^{1/k} \cdot \mathrm{Vol}(M)^{\frac{k-1}{k}}.$$

Here $d\mu$ *is the volume form of* M *and* $\mathrm{Vol}(M)$ *is the volume of* M*.*

Also, we will use the inequalities (c.f. [2])

$$(a_1 + a_2)^\theta \leq a_1^\theta + a_2^\theta, \quad 0 \leq \theta \leq 1, \tag{3.5}$$

$$(a_1 + a_2)^\theta \leq 2^{\theta-1}(a_1^\theta + a_2^\theta), \quad \theta \geq 1, \tag{3.6}$$

where a_1 and a_2 are any nonnegative numbers.

Theorem 3.4. *Suppose that* $k, n \geq 2$*, or,* $k = 1$ *and* $n > 2$*. Set*

$$Q_k = \frac{kn}{kn-(k+1)} = \frac{n}{n - \frac{k+1}{k}}. \tag{3.7}$$

Let M *be a compact* n*-dimensional hypersurface without boundary, which is smoothly embedded in* $\mathbb{R}^{n+1}$*. Then, for all nonnegative Lipschitz functions* v *on* M*, we have*

$$\|v\|^{k+1}_{L^{\frac{k+1}{k}Q_k}(M)} \leq A_{n,k}\left(\|\nabla v\|^{k+1}_{L^{\frac{k+1}{k}}(M)} + \|H\|^{n+k+1}_{L^{n+k+1}(M)}\|v\|^{k+1}_{L^{\frac{k+1}{k}}(M)}\right), \tag{3.8}$$

$$\leq \widehat{A}_{n,k}\left(\|\nabla v\|^{k+1}_{L^2(M)} + \|H\|^{n+k+1}_{L^{n+k+1}(M)}\|v\|^{k+1}_{L^2(M)}\right). \tag{3.9}$$

where $A_{n,k}$ *and* $\widehat{A}_{n,k}$ *are constants explicitly given by* ($c_{n,k} = c_n \cdot \frac{(k+1)(n-1)}{kn-(k+1)}$)

$$\begin{aligned} A_{n,k} &= 2^{\frac{(n-1)(k+1)(n+k+1)}{kn-(k+1)}}(2c_{n,k})^{n+k+1} \\ \widehat{A}_{n,k} &= A_{n,k} \cdot \mathrm{Vol}(M)^{\frac{k-1}{2(k+1)}}. \end{aligned}$$

Proof. The proof is quite similar to that given in [5]. The case that $k = 1$ and $n > 2$ has been proved in [5], hence we may assume that $k, n \geq 2$. Let

$$w = v^{\frac{(k+1)(n-1)}{kn-(k+1)}}.$$

Plugging it into (3.3),we have

$$\begin{aligned}\left(\int_M v^{\frac{n(k+1)}{kn-(k+1)}} d\mu\right)^{\frac{n-1}{n}} &\leq c_n \int_M \left(\frac{(k+1)(n-1)}{kn-(k+1)}|\nabla v| v^{\frac{n}{kn-(k+1)}} + |H| v^{\frac{(k+1)(n-1)}{kn-(k+1)}}\right) d\mu \\ &\leq c_{n,k}\left(\int_M |\nabla v| v^{\frac{n}{kn-(k+1)}} d\mu + \int_M |H| v^{\frac{(k+1)(n-1)}{kn-(k+1)}} d\mu\right),\end{aligned}$$

where

$$c_{n,k} := c_n \cdot \frac{(k+1)(n-1)}{kn-(k+1)} > c_n.$$

If we let $a_{n,k} = [c_{n,k}]^{\frac{kn-(k+1)}{n-1}} \cdot 2^{\frac{kn-k-n}{n-1}}$, then, using Hölder's inequality and the inequality (3.4), one concludes that (since $kn \geq k+n$)

$$\begin{aligned}&\left(\int_M v^{\frac{(k+1)n}{kn-(k+1)}} d\mu\right)^{\frac{kn-(k+1)}{n}} \\ \leq\ & [c_{n,k}]^{\frac{kn-(k+1)}{n-1}}\left(\int_M |\nabla v| v^{\frac{n}{kn-(k+1)}} d\mu + \int_M |H| v^{\frac{(k+1)(n-1)}{kn-(k+1)}} d\mu\right)^{\frac{kn-(k+1)}{n-1}} \\ \leq\ & a_{n,k}\left(\|\nabla v\|_{L^{\frac{k+1}{k}}(M)}^{\frac{kn-(k+1)}{n-1}} \|v\|_{L^{\frac{(k+1)n}{kn-(k+1)}}(M)}^{\frac{n}{n-1}} + \|H\|_{L^r(M)}^{\frac{kn-(k+1)}{n-1}} \|v\|_{L^{\frac{(k+1)(n-1)}{kn-(k+1)}s}(M)}^{k+1}\right)\end{aligned}$$

where r, s are positive real numbers satisfying $\frac{1}{r} + \frac{1}{s} = 1$. Recall Young's inequality

$$ab \leq \epsilon a^p + \epsilon^{-q/p} b^q,$$

where $a, b, \epsilon > 0$, $p, q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Putting

$$p = \frac{(k+1)(n-1)}{n}, \qquad q = \frac{(k+1)(n-1)}{kn-(k+1)}, \qquad \frac{p}{q} = \frac{kn-(k+1)}{n},$$

we derive that, for any $\epsilon > 0$,

$$\|\nabla v\|_{L^{\frac{k+1}{k}}(M)}^{\frac{kn-(k+1)}{n-1}} \|v\|_{L^{\frac{(k+1)n}{kn-(k+1)}}(M)}^{\frac{n}{n-1}} \leq \epsilon \|v\|_{L^{\frac{(k+1)n}{kn-(k+1)}}(M)}^{k+1} + \epsilon^{-\frac{n}{kn-(k+1)}} \|\nabla v\|_{L^{\frac{k+1}{k}}(M)}^{k+1}.$$

There is a natural way to find a suitable value of s, when we use the interpolation inequality to bound the first term appeared above using $L^{\frac{k+1}{k}}$-norm and $L^{\frac{(k+1)n}{kn-(k+1)}}$-norm. Suppose now that

$$\frac{kn-k-1}{kn-k} < 1 < s < \frac{n}{n-1}. \tag{3.10}$$

According to (3.10), we must have

$$\frac{k+1}{k} < \frac{(k+1)(n-1)}{kn-(k+1)} s < \frac{(k+1)n}{kn-(k+1)}.$$

Applying the interpolation inequality to our case gives

$$\|v\|_{L^{\frac{(k+1)(n-1)}{kn-(k+1)}s}(M)} \leq \delta \|v\|_{L^{\frac{(k+1)n}{kn-(k+1)}}(M)} + \delta^{-\mu} \|v\|_{L^{\frac{k+1}{k}}(M)}, \quad \delta > 0,$$

where the constant μ is determined by

$$\mu = \frac{\frac{k}{k+1} - \frac{kn-(k+1)}{(k+1)(n-1)s}}{\frac{kn-(k+1)}{(k+1)(n-1)s} - \frac{kn-(k+1)}{(k+1)n}} = \frac{n}{kn-(k+1)} \cdot \frac{k(n-1)(s-1)+1}{n-(n-1)s} := \mu_{n,k,s}.$$

Thus, together with Jensen's inequality, we yield

$$\begin{aligned} \|v\|^{k+1}_{L^{\frac{(k+1)n}{kn-(k+1)}}(M)} &\leq a_{n,k}\left(\epsilon\|v\|^{k+1}_{L^{\frac{(k+1)n}{kn-(k+1)}}(M)} + \epsilon^{-\frac{n}{kn-k-1}}\|\nabla v\|^{k+1}_{L^{\frac{k+1}{k}}(M)}\right. \\ &+ \left. 2^k\|H\|^{\frac{kn-(k+1)}{n-1}}_{L^r(M)}\left(\delta^{k+1}\|v\|^{k+1}_{L^{\frac{(k+1)n}{kn-(k+1)}}(M)} + \left(\delta^{k+1}\right)^{-\mu_{n,k,s}}\|v\|^{k+1}_{L^{\frac{k+1}{k}}(M)}\right)\right). \end{aligned}$$

Simplifying above implies that

$$\begin{aligned} &\left(1-\epsilon\cdot a_{n,k} - 2^k a_{n,k}\delta^{k+1}\|H\|^{\frac{kn-(k+1)}{n-1}}_{L^r(M)}\right)\|v\|^{k+1}_{L^{\frac{(k+1)n}{kn-(k+1)}}(M)} \\ \leq\ & a_{n,k}\epsilon^{-\frac{n}{kn-(k+1)}}\|\nabla v\|^{k+1}_{L^{\frac{k+1}{k}}(M)} + 2^k a_{n,k}\left(\delta^{k+1}\right)^{-\mu_{n.k.s}}\|H\|^{\frac{kn-(k+1)}{n-1}}_{L^r(M)}\|v\|^{k+1}_{L^{\frac{k+1}{k}}(M)}. \end{aligned}$$

Let (Here, we may assume that $\|H\|_{L^r(M)} \neq 0$; otherwise it is trivial.)

$$\epsilon = \frac{1}{2a_{n,k}}, \quad \delta^{k+1} = \frac{1}{2^{k+2}a_{n,k}}\|H\|^{-\frac{kn-(k+1)}{n-1}}_{L^r(M)}.$$

Therefore, we have (note that $\frac{1}{r}+\frac{1}{s}=1$)

$$\begin{aligned} \|v\|^{k+1}_{L^{\frac{(k+1)n}{kn-(k+1)}}(M)} &\leq 2(2a_{n,k})^{\frac{(k+1)(n-1)}{kn-(k+1)}}\|\nabla v\|^{k+1}_{L^{\frac{k+1}{k}}(M)} \\ &+ \left(2^{2+k}a_{n,k}\right)^{\frac{n-1}{kn-(k+1)}\cdot\frac{(k+1)r}{r-n}}\|H\|^{\frac{(k+1)r}{r-n}}_{L^r(M)}\|v\|^{k+1}_{L^{\frac{k+1}{k}}(M)}. \end{aligned}$$

The condition (3.9) turns out $r > n$. Setting

$$\frac{(k+1)r}{r-n} = r$$

gives us $r = n+k+1$ which is our required result. Plugging the explicit formula for $a_{n,k}$ in terms of $c_{n,k}$ into above and using Lemma 3.3, we obtain

$$\begin{aligned} \|v\|^{k+1}_{L^{\frac{k+1}{k}}(M)} &\leq 2(2c_{n,k})^{k+1}\|\nabla v\|^{k+1}_{L^{\frac{k+1}{k}}(M)} \\ &+ 2^{\frac{(n-1)(k+1)(n+k+1)}{kn-(k+1)}}(2c_{n,k})^{n+k+1}\|H\|^{n+k+1}_{L^{n+k+1}(M)}\|v\|^{k+1}_{L^{\frac{k+1}{k}}(M)}. \end{aligned}$$

Noting that the coefficient appeared in the first term is less than that in the second term, we obtain the inequality. □

Corollary 3.5. *Under the condition of Theorem 3.3, for any nonnegative Lipschitz functions v, we have*

$$\|v\|^2_{L^{2Q_k}(M)} \leq \widetilde{A}_{n,k}\left(\|v\|^{\frac{k-1}{k}}_{L^2(M)}\cdot\|\nabla v\|^{\frac{k+1}{k}}_{L^2(M)} + \left(\|H\|^{n+k+1}_{L^{n+k+1}(M)}\right)^{1/k}\|v\|^2_{L^2(M)}\right),$$

where the uniform constant $\widetilde{A}_{n,k}$ is given by

$$\widetilde{A}_{n,k} = A^{1/k}_{n,k}\cdot\left(\frac{2k}{k+1}\right)^{\frac{k+1}{k}}.$$

Proof. Replacing v by $v^{\frac{2k}{k+1}}$ in Theorem 3.3, we obtain

$$\begin{aligned}
\|v\|_{2Q_k}^{2k} &\leq A_{n,k}\left(\left\|\frac{2k}{k+1}\cdot v^{\frac{k-1}{k+1}}\cdot\nabla v\right\|_{L^{\frac{k+1}{k}}(M)}^{k+1} + \|H\|_{L^{n+k+1}(M)}^{n+k+1}\cdot\|v^{\frac{2k}{k+1}}\|_{L^{\frac{k+1}{k}}(M)}^{k+1}\right)\\
&= A_{n,k}\left(\left\|\left(\frac{2k}{k+1}\right)^{\frac{k+1}{k}} v^{\frac{k-1}{k}}(\nabla v)^{\frac{k+1}{k}}\right\|_{L^1(M)}^{k} + \|H\|_{L^{n+k+1}(M)}^{n+k+1}\cdot\|v\|_{L^2(M)}^{2k}\right)\\
&\leq A_{n,k}\left(\left(\frac{2k}{k+1}\right)^{k+1}\|v^{\frac{k-1}{k}}\|_{L^{\frac{2k}{k-1}}(M)}^{k}\|(\nabla v)^{\frac{k+1}{k}}\|_{L^{\frac{2k}{k+1}}(M)}^{k}\right)\\
&+ A_{n,k}\|H\|_{L^{n+k+1}(M)}^{n+k+1}\|v\|_{L^2(M)}^{2k}\\
&\leq A_{n,k}\left(\left(\frac{2k}{k+1}\right)^{k+1}\|v\|_{L^2(M)}^{k-1}\|\nabla v\|_{L^2(M)}^{k+1} + \|H\|_{L^{n+k+1}(M)}^{n+k+1}\|v\|_{L^2(M)}^{2k}\right).
\end{aligned}$$

Taking the kth root on both sides gives the required inequality. $\square$

Theorem 3.6. *Let n and k are integers bigger than or equal to 2. Consider the GMCF*

$$\frac{\partial}{\partial t}F(\cdot,t) = -f(H(\cdot,t))\nu(\cdot,t), \quad 0\leq t\leq T\leq T_{\max}<\infty,$$

where $f\in C^\infty(\Omega)$ is a smooth function over an open set $\Omega\subset\mathbb{R}$. Suppose that $f'(x)>0$ and $f(x)\cdot x\geq 0$ along the GMCF. For all nonnegative Lipschitz functions v, one has

$$\begin{aligned}
\|v\|_{L^\beta(M\times[0,T])}^{\beta} &\leq B_{n,k,T}\cdot\max_{0\leq t\leq T}\|v\|_{L^2(M_t)}^{\frac{(k+1)^2}{k^2 n}+\frac{k-1}{k}}\\
&\cdot\ \Big(\|\nabla_t v\|_{L^2(M\times[0,T])}^{\frac{k+1}{k}}\\
&+\ \max_{0\leq t\leq T}\|v\|_{L^2(M_t)}^{\frac{k+1}{k}}\cdot\left(\|H\|_{L^{n+k+1}(M\times[0,T])}^{n+k+1}\right)^{\frac{1}{k}}\Big),
\end{aligned}$$

where $B_{n,k,T}$ is the constant explicitly given by

$$B_{n,k,T} = \widetilde{A}_{n,k}\cdot\mathrm{Vol}(M)^{\frac{(k-1)(k+1)}{2k^2 n}}\cdot\max\left\{T^{\frac{k-1}{k}}, T^{\frac{k-1}{2k}}\right\}$$

and $\beta = 2+\frac{k+1}{k}\cdot\frac{k+1}{kn}>2$.

Proof. Setting $p=\frac{kn}{kn-(k+1)}$ and $q=\frac{kn}{k+1}$ in Hölder's inequality, we have

$$\begin{aligned}
\|v\|_{L^\beta(M\times[0,T])}^{\beta} &= \int_0^T dt\int_{M_t} v^2\cdot v^{\frac{k+1}{k}\cdot\frac{k+1}{kn}}d\mu(t)\\
&\leq \int_0^T dt\left(\int_{M_t} v^{2Q_k}d\mu(t)\right)^{1/Q_k}\left(\int_{M_t} v^{\frac{k+1}{k}}d\mu(t)\right)^{\frac{k+1}{kn}}\\
&= \max_{0\leq t\leq T}\|v\|_{L^{\frac{k+1}{k}}(M_t)}^{\frac{(k+1)^2}{k^2 n}}\cdot\int_0^T\|v\|_{L^{2Q_k}(M_t)}^2 dt.
\end{aligned}$$

The assumption $f(x)\cdot x\geq 0$ implies that

$$\frac{d}{dt}\mu(t) = -f(H(t))\cdot H(t)\mu(t)\leq 0,$$

consequently, the volume is deceasing along the GMCF. This fact combining with Lemma 3.2 gives

$$\begin{aligned}\max_{0\le t\le T}\|v\|^{\frac{(k+1)^2}{k^2 n}}_{L^{\frac{k+1}{k}}(M_t)} &\le \max_{0\le t\le T}\left(\|v\|_{L^2(M_t)}\cdot \mathrm{Vol}(M_t)^{\frac{k-1}{2(k+1)}}\right)^{\frac{(k+1)^2}{k^2 n}}\\ &\le \max_{0\le t\le T}\|v\|^{\frac{(k+1)^2}{k^2 n}}_{L^2(M_t)}\cdot \mathrm{Vol}(M)^{\frac{(k-1)(k+1)}{2k^2 n}}.\end{aligned}$$

On other hand, we have

$$\begin{aligned}\int_0^T \|v\|^2_{L^{2Q_k}(M_t)}dt &\le \widetilde{A}_{n,k}\int_0^T\left(\|v\|^{\frac{k-1}{k}}_{L^2(M_t)}\cdot\|\nabla_t v\|^{\frac{k+1}{k}}_{L^2(M_t)}\right.\\ &+ \left.\|v\|^2_{L^2(M_t)}\left(\|H\|^{n+k+1}_{L^{n+k+1}(M_t)}\right)^{\frac{1}{k}}\right)dt\\ &\le \widetilde{A}_{n,k}\cdot\max_{0\le t\le T}\|v\|^{\frac{k-1}{k}}_{L^2(M_t)}\cdot\int_0^T\|\nabla_t v\|^{\frac{k+1}{k}}_{L^2(M_t)}dt\\ &+ \widetilde{A}_{n,k}\cdot\max_{0\le t\le T}\|v\|^2_{L^2(M_t)}\cdot\int_0^T\left(\|H\|^{n+k+1}_{L^{n+k+1}(M_t)}\right)^{1/k}dt.\end{aligned}$$

From Lemma 3.2, we obtain

$$\begin{aligned}\int_0^T\left(\|H\|^{n+k+1}_{L^{n+k+1}(M_t)}\right)^{1/k}dt &\le \left(\int_0^T\|H\|^{n+k+1}_{L^{n+k+1}(M_t)}dt\right)^{1/k}T^{\frac{k-1}{k}},\\ &= \left(\|H\|^{n+k+1}_{L^{n+k+1}(M\times[0,T])}\right)^{1/k}\cdot T^{\frac{k-1}{k}},\\ \int_0^T\|\nabla v\|^{\frac{k+1}{k}}_{L^2(M_t)}dt &= \int_0^T\left(\|\nabla_t v\|^2_{L^2(M_t)}\right)^{\frac{1}{2k/(k+1)}}dt\\ &\le \|\nabla_t v\|^{\frac{k+1}{k}}_{L^2(M\times[0,T])}\cdot T^{\frac{k-1}{2k}}.\end{aligned}$$

Plugging it into above inequality, one yields

$$\begin{aligned}\|v\|^\beta_{L^\beta(M\times[0,T])} &\le \max_{0\le t\le T}\|v\|^{\frac{(k+1)^2}{k^2 n}}_{L^2(M_t)}\cdot(\mathrm{Vol}(M))^{\frac{(k-1)(k+1)}{2k^2 n}}\cdot\widetilde{A}_{n,k}\\ &\cdot \max_{0\le t\le T}\|v\|^{\frac{k-1}{k}}_{L^2(M_t)}\cdot\max\left\{T^{\frac{k-1}{k}},T^{\frac{k-1}{2k}}\right\}\\ &\cdot \left(\|\nabla_t v\|^{\frac{k+1}{k}}_{L^2(M_t)}\right.\\ &+ \left.\max_{0\le t\le T}\|v\|^{\frac{k+1}{k}}_{L^2(M_t)}\left(\|H\|^{n+k+1}_{L^{n+k+1}(M\times[0,T])}\right)^{1/k}\right),\end{aligned}$$

which is the required result. □

Remark 3.7. *If $k=1$, then $\frac{k+1}{k}=2$; hence we do not need to use Lemma 3.2 to control the terms by L^2-norm and carefully checking the proof gives $B_{n,1,T}=A_{n,1}$, which is the constant derived in* [5].

4. Moser iteration for the H^k mean curvature flow

In this section we generalize Lemma 4.1 in [5] to the GMCF, in particular, to the H^k mean curvature flow. The proof is similar to that given in [5], but it doesn't directly follow words by words from [5] since the differential inequality now involves

an extra term $f''(v)|\nabla v|^2$. When $f(x) = x^k$ and $k = 1$, that is, the classical mean curvature flow, this term automatically vanishes. Since the mean curvature $H(t)$ along the generalized mean curvature flow satisfies

$$\frac{\partial}{\partial t}H(t) = f'(H(t))\Delta_t H(t) + f(H(t))|A(t)|^2_{g(t)} + f''(H(t))|\nabla_t H(t)|^2_{g(t)},$$

we should study the differential inequality

$$\left(\frac{\partial}{\partial t} - \Delta_{f,t}\right) v \leq G \cdot f(v) + f''(v)|\nabla_t v|^2_{g(t)}, \quad v \geq 0, \quad G \in L^q(M \times [0,T]). \tag{4.1}$$

Let $\eta(x,t)$ be any smooth function on $M \times [0,T]$ with the property that $\eta(x,0) = 0$ for all $x \in M$.

Later, we will chose $\eta(x,t)$ to be a smooth function only relative to the variable t, satisfying the above property, and $f(x) = x^k$.

Theorem 4.1. *Suppose that the integers n and k are greater than or equal to 2. Consider the GMCF*

$$\frac{\partial}{\partial t}F(\cdot,t) = -f(H(\cdot,t))\nu(\cdot,t), \quad 0 \leq t \leq T \leq T_{\max} < \infty.$$

Suppose that $f \in C^\infty(\Omega)$ for an open set $\Omega \subset \mathbb{R}$, and that v is a smooth function on $M \times [0,T]$ such that its image is contained in Ω. Consider the differential inequality

$$\left(\frac{\partial}{\partial t} - \Delta_{f,t}\right) v \leq G \cdot f(v) + f''(v)|\nabla_t v|^2, \quad v \geq 0, \quad G \in L^q(M \times [0,T]). \tag{4.2}$$

Let

$$\begin{aligned} C_{0,q} &= \|f'(v)G\|_{L^q(M\times[0,T])}, \\ C_1 &= \left(1 + \|H\|^{n+k+1}_{L^{n+k+1}(M\times[0,T])}\right)^{\frac{1}{k}}, \end{aligned}$$

and also let

$$\gamma = 2 + \frac{(k+1)^2}{k^2 n}.$$

We denote by $\mathcal{S}$ the set of all functions $f \in C^\infty(\Omega)$, where $\Omega \subset \mathbb{R}$ is the domain of f, satisfying

(i) *f satisfies the differential inequality (4.2),*
(ii) *$f'(x) > 0$ for all $x \in \Omega$,*
(iii) *$f(x) \geq 0$ whenever $x \geq 0$,*
(iv) *$f(H(t))H(t) \geq 0$ along the GMCF.*
(v) *$f'(v) \geq C_2 > 0$ on $M \times [0,T]$ for some uniform constant C_2.*

For any $\beta \geq 2$ and $q > \frac{\gamma}{\gamma-2}$, there exists a positive constant $C_{n,k,T}(C_{0,q}, C_1, \beta, q)$, depending only on n, k, T, β, q, $C_{0,q}$, C_1, and $\mathrm{Vol}(M)$, such that, for any $f \in \mathcal{S}$,

$$\begin{aligned} & \|\eta^2 f^\beta(v)\|_{L^{\gamma/2}(M\times[0,T])} \\ \leq\ & C_{n,k,T}(C_{0,q}, C_1, \beta, q) \left\| f^\beta(v)\left[\eta^2 + 2\eta\left(\frac{\partial}{\partial t} - f'(v)\Delta_t\right)\eta \right.\right. \\ +\ & \left.\left. \left(\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)} + \frac{8\beta^2 - 2\beta + 2}{\beta(\beta-1)}f'(v)\right)|\nabla_t \eta|^2_{g(t)}\right]\right\|_{L^1(M\times[0,T])} \end{aligned}$$

where

$$\begin{aligned} & C_{n,k,T}(C_{0,q}, C_1, \beta, q) \\ = \quad & \frac{\beta}{\beta-1}\max\left\{2(B_{n,k,T}C_1)^{2/\gamma}, \left(2C_{0,q}\frac{\beta^2}{\beta-1}(B_{n,k,T}C_1)^{2/\gamma}\right)^{1+\nu}\right\}, \end{aligned}$$

$\nu = \frac{\gamma}{(\gamma-2)q-\gamma}$, *and* η *is any smooth function on* $M \times [0,T]$ *with the property that* $\eta(x,0) = 0$ *for all* $x \in M$. *In particular, if* $f'(v)G \in L^\infty(M \times [0,T])$, *then, letting* $q \to \infty$, *we have*

$$\begin{aligned} & C_{n,k,T}(C_{0,\infty}, C_1, \beta, \infty) \\ = \quad & \frac{2\beta}{\beta-1}\max\left\{1, \frac{C_{0,\infty}\beta^2}{\beta-1}\right\}(\widetilde{B}_{n,k,T}C_1)^{2/\gamma} \\ \leq \quad & \left[8\max\{1, C_{0,\infty}\}\widetilde{B}^{2/\gamma}_{n,k,T}\right]\beta C_1^{2/\gamma}, \end{aligned}$$

where

$$\begin{aligned} \widetilde{B}_{n,k,T} &= B_{n,k,T}\cdot\max\left\{\left(\frac{1}{C_2}\right)^{\frac{k+1}{2k}}, 1\right\}, \\ C_{0,\infty} &= \|f'(v)G\|_{L^\infty(M\times[0,T])}, \end{aligned}$$

since $\frac{\beta}{\beta-1} \leq 2$; *in this case, we obtain*

$$\begin{aligned} & \|\eta^2 f^\beta(v)\|_{L^{\gamma/2}(M\times[0,T])} \\ \leq \quad & D_{n,k,T}\beta C_1^{2/\gamma}\left\| f^\beta(v)\left[\eta^2 + 2\eta\left(\frac{\partial}{\partial t} - f'(v)\Delta_t\right)\eta\right.\right. \\ + \quad & \left.\left.\left(\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)} + \frac{8\beta^2-2\beta+2}{\beta(\beta-1)}f'(v)\right)|\nabla_t\eta|^2_{g(t)}\right]\right\|_{L^1(M\times[0,T])}, \end{aligned}$$

where $D_{n,k,T} = 8\max\{1, C_{0,\infty}\}\widetilde{B}^{2/\gamma}_{n,k,T}$.

Remark 4.2. *The set* $\mathcal{S}$, *in general, may not be empty. For example, let* $v(\cdot,t) = H(\cdot,t) \geq 0$ *and suppose that* $f(x) = x^k$, $\Omega = \mathbb{R}^+$, *and* $f'(H(t)) \geq C_2 > 0$ *along the GMCF; we immediately see that the conditions (ii) (iii), and (v) are satisfied. For (iv),*

$$f(H(t))H(t) = H^{k+1}(t) = H^{k-1}(t)\cdot H^2(t) \geq 0.$$

This will be applied to our case.

Proof. Applying the test function $\eta^2 f'(v)f^{\beta-1}(v)$ to our differential inequality (4.1), for any $s \in [0,T]$, we have

$$\begin{aligned} & \int_0^s\int_{M_t}(-\Delta_{f,t}v)\eta^2 f'(v)f^{\beta-1}(v)d\mu(t)dt \\ + \quad & \int_0^s\int_{M_t}\frac{\partial v}{\partial t}\eta^2 f'(v)f^{\beta-1}(v)d\mu(t)dt \\ \leq \quad & \int_0^s\int_{M_t}|G|\eta^2 f'(v)f^\beta(v)d\mu(t)dt \\ + \quad & \int_0^s\int_{M_t}\eta^2 f'(v)f''(v)f^{\beta-1}(v)|\nabla_t v|^2_{g(t)}d\mu(t)dt. \end{aligned}$$

Integrating by parts gives

$$
\begin{aligned}
& \int_{M_t}(-\Delta_{f,t}v)\eta^2 f'(v)f^{\beta-1}(v)d\mu(t)dt \ = \ \int_{M_t}(-\Delta_t v)\eta^2(f'(v))^2 f^{\beta-1}(v)d\mu(t) \\
= \ & \int_{M_t}\langle\nabla_t v, \nabla_t(\eta^2(f'(v))^2 f^{\beta-1}(v))\rangle_{g(t)}d\mu(t) \\
= \ & \int_{M_t}\langle\nabla_t v, 2\nabla_t\eta\cdot\eta(f'(v))^2 f^{\beta-1}(v)\rangle_{g(t)}d\mu(t) \\
+ \ & \int_{M_t}\langle\nabla_t v, \eta^2(2f'(v)f''(v)f^{\beta-1}(v)\nabla_t v + (f'(v))^3(\beta-1)f^{\beta-2}(v)\nabla_t v)\rangle_{g(t)}d\mu(t) \\
= \ & 2\int_{M_t}\langle\nabla_t v, \nabla_t\eta\rangle_{g(t)}\eta(f'(v))^2 f^{\beta-1}(v)d\mu(t) \\
+ \ & \int_{M_t}\eta^2[2f'(v)f''(v)f^{\beta-1}(v) + (\beta-1)(f'(v))^3 f^{\beta-2}(v)]|\nabla_t v|^2_{g(t)}d\mu(t).
\end{aligned}
$$

Recall the evolution equation for volume form

$$
\frac{\partial}{\partial r}d\mu(t) = -f(H(t))\cdot H(t)\cdot d\mu(t).
$$

Hence

$$
\begin{aligned}
& \int_0^s\int_{M_t}\frac{\partial v}{\partial t}\cdot\eta^2\cdot f'(v)f^{\beta-1}(v)d\mu(t)dt \\
= \ & \frac{1}{\beta}\int_0^s\int_{M_t}\frac{\partial(f^\beta(v))}{\partial t}\eta^2 d\mu(t)dt \\
= \ & \frac{1}{\beta}\int_{M_t}f^\beta(v)\eta^2 d\mu(t)\Big|_0^s - \frac{1}{\beta}\int_0^s\int_{M_t}f^\beta(v)\frac{\partial}{\partial t}(\eta^2 d\mu(t))dt \\
= \ & \frac{1}{\beta}\int_{M_s}f^\beta(v)\eta^2 d\mu(s) - \frac{1}{\beta}\int_0^s\int_{M_t}f^\beta(v)\left[2\eta\frac{\partial\eta}{\partial t} - \eta^2 f(H(t))H(t)\right]d\mu(t)dt.
\end{aligned}
$$

Combining these formulas and the assumption (iii), we conclude that

$$
\begin{aligned}
& \int_0^s\int_{M_t}\Big[2\langle\nabla_t v, \nabla_t\eta\rangle_{g(t)}\eta(f'(v))^2 f^{\beta-1}(v) \\
+ \ & (2\eta^2 f'(v)f''(v)f^{\beta-1}(v) + (\beta-1)\eta^2(f'(v))^3 f^{\beta-1}(v))|\nabla_t v|^2_{g(t)}\Big]d\mu(t)dt \\
+ \ & \frac{1}{\beta}\int_{M_s}f^\beta(v)\eta^2 d\mu(s) \\
\leq \ & \frac{1}{\beta}\int_0^s\int_{M_t}f^\beta(v)\left[2\eta\frac{\partial\eta}{\partial t} - \eta^2 f(H(t))H(t)\right]d\mu(t)dt \\
+ \ & \int_0^s\int_{M_t}|G|\eta^2 f'(v)f^\beta(v)d\mu(t)dt \\
+ \ & \int_0^s\int_{M_t}\eta^2 f'(v)f''(v)f^{\beta-1}(v)|\nabla_t v|^2_{g(t)}d\mu(t)dt \\
\leq \ & \frac{1}{\beta}\int_0^s\int_{M_t}f^\beta(v)2\eta\frac{\partial\eta}{\partial t}d\mu(t)dt + \int_0^s\int_{M_t}|G|\eta^2 f'(v)f^\beta(v)d\mu(t)dt \\
+ \ & \int_0^s\int_{M_t}\eta^2 f'(v)f''(v)f^{\beta-1}(v)|\nabla_t v|^2_{g(t)}d\mu(t)dt.
\end{aligned}
$$

Since

$$\begin{aligned}
&\frac{1}{\beta}\int_0^s\int_{M_t} f^\beta(v)2\eta\frac{\partial\eta}{\partial t}d\mu(t)dt\\
= & \frac{1}{\beta}\int_0^s\int_{M_t}\left[f^\beta(v)2\eta\left(\frac{\partial}{\partial t}-f'(v)\Delta_t\right)\eta+f^\beta(v)f'(v)2\eta\Delta_t\eta\right]d\mu(t)dt\\
= & \frac{1}{\beta}\int_0^s\int_{M_t}\left[f^\beta(v)2\eta\left(\frac{\partial}{\partial t}-f'(v)\Delta_t\right)\eta-2\langle\nabla_t(f^\beta(v)f'(v)\eta),\nabla_t\eta\rangle_{g(t)}\right]d\mu(t)dt\\
= & \frac{1}{\beta}\int_0^s\int_{M_t}\left[f^\beta(v)2\eta\left(\frac{\partial}{\partial t}-f'(v)\Delta_t\right)\eta-2\langle\beta f^{\beta-1}(v)(f'(v))^2\eta\nabla_t v,\nabla_t\eta\rangle_{g(t)}\right.\\
- & \left.2\langle f^\beta(v)(\eta f''(v)\nabla_t v+f'(v)\nabla_t\eta),\nabla_t\eta\rangle_{g(t)}\right]d\mu(t)dt\\
= & \frac{1}{\beta}\int_0^s\int_{M_t} f^\beta(v)\left[2\eta\left(\frac{\partial}{\partial t}-f'(v)\Delta_t\right)\eta-2f'(v)|\nabla_t\eta|^2_{g(t)}\right]d\mu(t)dt\\
- & \frac{2}{\beta}\int_0^s\int_{M_t}\eta\left[\beta f^{\beta-1}(v)(f'(v))^2+f^\beta(v)f''(v)\right]\langle\nabla_t v,\nabla_t\eta\rangle_{g(t)}d\mu(t)dt
\end{aligned}$$

it follows that

$$\begin{aligned}
&4\int_0^s\int_{M_t}\eta(f'(v))^2f^{\beta-1}(v)\langle\nabla_t v,\nabla_t\eta\rangle_{g(t)}d\mu(t)dt \ + \ \frac{1}{\beta}\int_{M_s}f^\beta(v)\eta^2d\mu(s)\\
+ & \int_0^s\int_{M_t}[(\beta-1)(f'(v))^3+f(v)f'(v)f''(v)]\eta^2f^{\beta-2}(v)|\nabla_t v|^2_{g(t)}d\mu(t)dt\\
\leq & \frac{1}{\beta}\int_0^s\int_{M_t} f^\beta(v)\left[2\eta\left(\frac{\partial}{\partial t}-f'(v)\Delta_t\right)\eta-2f'(v)|\nabla_t\eta|^2_{g(t)}\right]d\mu(t)dt\\
+ & \int_0^s\int_{M_t}|G|\eta^2f'(v)f^\beta(v)d\mu(t)dt\\
- & \frac{2}{\beta}\int_0^s\int_{M_t}\eta f''(v)f^\beta(v)\langle\nabla_t v,\nabla_t\eta\rangle_{g(t)}d\mu(t)dt.
\end{aligned}$$

The Cauchy-Schwartz inequality gives (where $\epsilon>0$)

$$\begin{aligned}
&4\int_0^s\int_{M_t}\langle\nabla_t v,\nabla_t\eta\rangle_{g(t)}\eta(f'(v))^2f^{\beta-1}(v)d\mu(t)dt\\
\geq & -2\epsilon^2\int_0^s\int_{M_t}\eta^2(f'(v))^3f^{\beta-2}(v)|\nabla_t v|^2_{g(t)}d\mu(t)dt\\
- & \frac{2}{\epsilon^2}\int_0^s\int_{M_t}f'(v)f^\beta(v)|\nabla_t\eta|^2_{g(t)}d\mu(t)dt,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{2}{\beta}\int_0^s\int_{M_t}\eta f''(v)f^\beta(v)\langle\nabla_t v,\nabla_t\eta\rangle_{g(t)}d\mu(t)dt\\
\geq & -\int_0^s\int_{M_t}f(v)f'(v)f''(v)f^{\beta-2}(v)\eta^2|\nabla_t v|^2_{g(t)}d\mu(t)dt\\
- & \frac{1}{\beta^2}\int_0^s\int_{M_t}\frac{f(v)f''(v)}{f'(v)}f^\beta(v)|\nabla_t\eta|^2_{g(t)}d\mu(t)dt.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
& \int_0^s \int_{M_t} [(\beta-1-2\epsilon^2)f'(v)]\eta^2 f^{\beta-2}(v)(f'(v))^2|\nabla_t v|^2_{g(t)}d\mu(t)dt \\
+ & \frac{1}{\beta}\int_{M_s} f^\beta(v)\eta^2 d\mu(s) \\
\leq & \frac{1}{\beta}\int_0^s \int_{M_t} f^\beta(v)\left[2\eta\left(\frac{\partial}{\partial t} - f'(v)\Delta_t\right)\eta\right. \\
+ & \left.\left(\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)} - 2f'(v) + \frac{2\beta}{\epsilon^2}f'(v)\right)|\nabla_t\eta|^2_{g(t)}\right]d\mu(t)dt \\
+ & \int_0^s \int_{M_t} |G|\eta^2 f'(v) f^\beta(v)d\mu(t)dt.
\end{aligned}$$

Note that

$$|\nabla_t(f^{\beta/2}(v))|^2_{g(t)} = \frac{\beta^2}{4}f^{\beta-2}(v)(f'(v))^2|\nabla_t v|^2_{g(t)}.$$

If we choose $\beta - 1 = 4\epsilon^2$, then the above inequality gives us

$$\begin{aligned}
& \frac{2(\beta-1)}{\beta}\int_0^s \int_{M_t} f'(v)\eta^2|\nabla_t(f^{\beta/2}(v))|^2_{g(t)}d\mu(t)dt + \int_{M_s} f^\beta(v)\eta^2 d\mu(s) \\
\leq & \int_0^s \int_{M_t} f^\beta(v)\left[2\eta\left(\frac{\partial}{\partial t} - f'(v)\Delta_t\right)\eta\right. \\
+ & \left.\left(\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)} - 2f'(v) + \frac{8\beta}{\beta-1}f'(v)\right)|\nabla_t\eta|^2_{g(t)}\right]d\mu(t)dt \\
+ & \beta\int_0^s \int_{M_t} |G|\eta^2 f'(v) f^\beta(v)d\mu(t)dt.
\end{aligned}$$

Recall that

$$\begin{aligned}
|\nabla_t(\eta f^{\beta/2}(v))|^2_{g(t)} &= |\nabla_t\eta\cdot f^{\beta/2}(v) + \eta\nabla_t(f^{\beta/2}(v))|^2_{g(t)} \\
&\leq 2\eta^2|\nabla_t(f^{\beta/2}(v))|^2_{g(t)} + 2f^\beta(v)\cdot|\nabla_t\eta|^2_{g(t)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& C_2\int_0^s \int_{M_t} |\nabla_t(\eta f^{\beta/2})|^2_{g(t)}d\mu(t)dt + \int_{M_s} f^\beta(v)\eta^2 d\mu(t) \\
\leq & \frac{\beta}{\beta-1}\int_0^s \int_{M_t} f^\beta(v)\left[2\eta\left(\frac{\partial}{\partial t} - f'(v)\Delta_t\right)\eta\right. \\
+ & \left.\left(\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)} + \frac{8\beta^2-2\beta+2}{\beta(\beta-1)}f'(v)\right)|\nabla_t\eta|^2_{g(t)}\right]d\mu(t)dt \\
+ & \frac{\beta^2}{\beta-1}\int_0^s \int_{M_t} |G|\eta^2 f'(v) f^\beta(v)d\mu(t)dt \\
\leq & \frac{\beta}{\beta-1}\int_0^s \int_{M_t} f^\beta(v)\left[2\eta\left(\frac{\partial}{\partial t} - f'(v)\Delta_t\right)\eta\right. \\
+ & \left.\left(\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)} + \frac{8\beta^2-2\beta+2}{\beta(\beta-1)}f'(v)\right)|\nabla_t\eta|^2_{g(t)}\right]d\mu(t)dt \\
+ & \frac{\beta^2}{\beta-1}\|f'(v)G\|_{L^q(M\times[0,T])}\cdot\|\eta^2 f^\beta\|_{L^{\frac{q}{q-1}}(M\times[0,T])} := A.
\end{aligned}$$

(In the following we also use the notion Λ which is the first term of A.) It gives us, for any s,

$$\begin{aligned}\|\eta f^{\beta/2}(v)\|_{L^2(M_s)} &\leq A^{1/2},\\ \|\nabla_t(\eta f^{\beta/2}(v))\|_{L^2(M\times[0,T])} &\leq \left(\frac{A}{C_2}\right)^{1/2}.\end{aligned}$$

Using Theorem 3.6, one has

$$\begin{aligned}\|\eta f^{\beta/2}(v)\|^{\gamma}_{L^{\gamma}(M\times[0,T])} &\leq B_{n,k,T}\cdot\max_{0\leq s\leq T}\|\eta f^{\beta/2}(v)\|^{\frac{(k+1)^2}{k^2n}+\frac{k-1}{k}}_{L^2(M_s)}\\ &\cdot \left(\|\nabla_t(\eta f^{\beta/2}(v))\|^{\frac{k+1}{k}}_{L^2(M\times[0,T])}\right.\\ &+ \left.\max_{0\leq s\leq T}\|\eta f^{\beta/2}(v)\|^{\frac{k+1}{k}}_{L^2(M_s)}\cdot\left(\|H\|^{n+k+1}_{L^{n+k+1}(M\times[0,T])}\right)^{1/k}\right)\\ &\leq B_{n,k,T}\cdot\max\left\{\left(\frac{1}{C_2}\right)^{\frac{k+1}{2k}},1\right\}\cdot A^{\frac{(k+1)^2}{2k^2n}+1}\\ &\cdot \left[1+\left(\|H\|^{n+k+1}_{L^{n+k+1}(M\times[0,T])}\right)^{1/k}\right],\\ &= [\widetilde{B}_{n,k,T}C_1]\cdot A^{\frac{(k+1)^2}{2k^2n}+1},\end{aligned}$$

where $\gamma = 2+\frac{k+1}{k}\cdot\frac{k+1}{kn}$. Moreover,

$$\begin{aligned}\|\eta^2 f^{\beta}\|_{L^{\gamma/2}(M\times[0,T])} &= \left(\|\eta f^{\beta/2}\|^{\gamma}_{L^{\gamma}(M\times[0,T])}\right)^{2/\gamma}\\ &\leq A\cdot(\widetilde{B}_{n,k,T}C_1)^{2/\gamma}\\ &= (\widetilde{B}_{n,k,T}C_1)^{2/\gamma}\left(\Lambda+\frac{\beta^2}{\beta-1}C_0\|\eta^2 f^{\beta}\|_{L^{\frac{q}{q-1}}(M\times[0,T])}\right),\end{aligned}$$

where $q>\frac{\gamma}{\gamma-2}$. Noting that

$$1<\frac{q}{q-1}<\frac{\gamma}{2}$$

and using the interpolation inequality, one gets

$$\|\eta^2 f^{\beta}\|_{L^{\frac{q}{q-1}}(M\times[0,T])}\leq \epsilon\|\eta^2 f^{\beta}\|_{L^{\gamma/2}(M\times[0,T])}+\epsilon^{-\nu}\|\eta^2 f^{\beta}\|_{L^1(M\times[0,T])},$$

where the constant ν is defined by

$$\nu = \frac{1-\frac{q-1}{q}}{\frac{q-1}{q}-\frac{2}{\gamma}} = \frac{\gamma}{(\gamma-2)q-\gamma}.$$

Therefore,

$$\begin{aligned}&\|\eta^2 f^{\beta}\|_{L^{\gamma/2}(M\times[0,T])}\\ \leq\ &\left[(\widetilde{B}_{n,k,T}C_1)^{2/\gamma}\cdot\frac{\beta^2}{\beta-1}C_{0,q}\epsilon\right]\|\eta^2 f^{\beta}\|_{L^{\gamma/2}(M\times[0,T])}\\ +\ &(\widetilde{B}_{n,k,T}C_1)^{2/\gamma}\left(\Lambda+\frac{\beta^2}{\beta-1}C_{0,q}\epsilon^{-\nu}\|\eta^2 f^{\beta}(v)\|_{L^1(M\times[0,T])}\right).\end{aligned}$$

If we chose $(\widetilde{B}_{n,k,T}C_1)^{2/\gamma}\cdot\frac{\beta^2}{\beta-1}\cdot C_{0,q}\epsilon=\frac{1}{2}$, then

$$\begin{aligned}
& \|\eta^2 f^\beta(v)\|_{L^{\gamma/2}(M\times[0,T])} \\
\leq\ & 2(\widetilde{B}_{n,k,T}C_1)^{2/\gamma}\Lambda \\
+\ & \left(2C_{0,q}\cdot\frac{\beta^2}{\beta-1}(\widetilde{B}_{n,k,T}C_1)^{\frac{2}{\gamma}}\right)^{1+\nu}\|\eta^2 f^\beta(v)\|_{L^1(M\times[0,T])} \\
\leq\ & \max\left\{2(\widetilde{B}_{n,k,T}C_1)^{2/\gamma},\left(2C_{0,q}\cdot\frac{\beta^2}{\beta-1}(\widetilde{B}_{n,k,T}C_1)^{\frac{2}{\gamma}}\right)^{1+\nu}\right\} \\
\cdot\ & \left(\Lambda+\|\eta^2 f^\beta(v)\|_{L^1(M\times[0,T])}\right) \\
:=\ & \widetilde{C}_{n,k,T}(C_{0,q},C_1,\beta,q)\cdot\left(\Lambda+\|\eta^2 f^\beta(v)\|_{L^1(M\times[0,T])}\right),
\end{aligned}$$

where $\widetilde{C}_{n,k,T}(C_{0,q},C_1,\beta,q)$ is the constant depending only on n,k,T,β,q, $C_{0,q}$, C_1, and $\mathrm{Vol}(M)$. From the definition of A and noting that $1<\frac{\beta}{\beta-1}\leq 2$, one yields

$$\begin{aligned}
& \|\eta^2 f^\beta(v)\|_{L^{\gamma/2}(M\times[0,T])} \\
\leq\ & \widetilde{C}_{n,k,T}(C_{0,q},C_1,\beta,q)\left(\frac{\beta}{\beta-1}\int_0^s\int_{M_t}f^\beta(v)\left[2\eta\left(\frac{\partial}{\partial t}-f'(v)\Delta_t\right)\eta\right.\right. \\
+\ & \left.\left(\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)}+\frac{8\beta^2-2\beta+2}{\beta(\beta-1)}f'(v)\right)|\nabla_t\eta|^2_{g(t)}\right]d\mu(t)dt \\
+\ & \left.\int_0^s\int_{M_t}f^\beta(v)\eta^2 d\mu(t)dt\right) \\
\leq\ & C_{n,k,T}(C_{0,q},C_1,\beta,q)\int_0^s\int_{M_t}f^\beta(v)\left[\eta^2+2\eta\left(\frac{\partial}{\partial t}-f'(v)\Delta_t\right)\eta\right. \\
+\ & \left.\left(\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)}+\frac{8\beta^2-2\beta+2}{\beta(\beta+1)}f'(v)\right)|\nabla_t\eta|^2_{g(t)}\right]d\mu(t)dt \\
=\ & C_{n,k,T}(C_{0,q},C_1,\beta,q)\left\|f^\beta(v)\left[\eta^2+2\eta\left(\frac{\partial}{\partial t}-f'(v)\Delta_t\right)\eta\right.\right. \\
+\ & \left.\left.\left(\frac{1}{\beta}\frac{f(v)f''(v)}{f'(v)}+\frac{8\beta^2-2\beta+2}{\beta(\beta-1)}f'(v)\right)|\nabla_t\eta|^2_{g(t)}\right]\right\|_{L^1(M\times[0,T])},
\end{aligned}$$

which is our required result. □

Taking some special smooth function and using the Moser iteration, we can prove that the L^∞-norm of v over a smaller domain is bounded by some L^β-norm of v over the whole manifold $M\times[0,T]$.

Corollary 4.3. *Suppose that the integers n and k are greater than or equal to 2. Consider the GMCF*

$$\frac{\partial}{\partial t}F(\cdot,t)=-f(H(\cdot,t))\nu(\cdot,t),\quad 0\leq t\leq T\leq T_{\max}<\infty.$$

Suppose that $f\in C^\infty(\Omega)$ for an open set $\Omega\subset\mathbb{R}$, and that v is a smooth function on $M\times[0,T]$ such that its image is contained in Ω. Consider the differential inequality

$$\left(\frac{\partial}{\partial t}-\Delta_{f,t}\right)v\leq G\cdot f(v)+f''(v)|\nabla_t v|^2,\quad v\geq 0,\quad G\in L^q(M\times[0,T]).\tag{4.3}$$

Let

$$C_{0,\infty} = \|f'(v)G\|_{L^\infty(M\times[0,T])}, \quad C_1 = \left(1 + \|H\|^{n+k+1}_{L^{n+k+1}(M\times[0,T])}\right)^{\frac{1}{k}},$$

and also let

$$\gamma = 2 + \frac{(k+1)^2}{k^2 n}.$$

We denote by $\mathcal{S}$ the set of all functions $f \in C^\infty(\Omega)$, where $\Omega \subset \mathbb{R}$ is the domain of f, satisfying

(i) *f satisfies the differential inequality (4.3),*
(ii) *$f'(x) > 0$ for all $x \in \Omega$,*
(iii) *$f(x) \geq 0$ whenever $x \geq 0$,*
(iv) *$f(H(t))H(t) \geq 0$ along the GMCF.*
(v) *$f'(v) \geq C_2 > 0$ on $M \times [0,T]$ for some uniform constant C_2.*

There exists an uniform constant $C_n > 0$, depending only on n, such that for any $\beta \geq 2$ and $f \in \mathcal{S}$ we have

$$\|f(v)\|_{L^\infty\left(M\times\left[\frac{T}{2},T\right]\right)} \leq E_{n,k,T}(\beta) \cdot C_1^{\frac{1}{\beta}\frac{2}{\gamma-2}} \cdot \|f(v)\|_{L^\beta(M\times[0,T])},$$

where

$$E_{n,k,T}(\beta) = (D_{n,k,T}C_n\beta)^{\frac{1}{\beta}\frac{\gamma}{\gamma-2}} \cdot \left(\frac{\gamma}{2}\right)^{\frac{1}{\beta}\frac{2\gamma}{(\gamma-2)^2}} \cdot 4^{\frac{1}{\beta}\frac{\gamma^2}{(\gamma-2)^2}}.$$

Proof. Consider an increasing sequence of times t_i defined by

$$t_i = \frac{T}{2}\left(1 - \frac{1}{4^i}\right), \quad i = 0, 1, 2, \cdots.$$

Consider a sequence of smooth function $\eta_l(t)$ satisfying the following properties

$$\eta_i|_{[t_i,T]} \equiv 1, \quad \eta_i|_{[0,t_{i-1}]} \equiv 0, \quad 0 \leq \eta \leq 1, \quad |\eta_i'| \leq C_n 4^i.$$

For convenience, we denote by I_i the interval $[t_i, T]$. Since $\|f'(v)G\|_{L^\infty(M\times[0,T])}$ exists, letting $\gamma \to \infty$, we have

$$\|f^\beta(v)\|_{L^{\gamma/2}(M\times I_i)} \leq [D_{n,k,T} \cdot C_n \cdot 4^i] \cdot \beta \cdot C_1^{2/\gamma}\|f^\beta(v)\|_{L^1(M\times I_{i-1})}.$$

For a moment we put $C = D_{n,k,T}C_n$, $\|\cdot\|_{p,i} = \|\cdot\|_{L^p(M\times I_i)}$, $\widehat{\gamma} = \gamma/2$, and $w = f(v)$. Hence

$$\|w^\beta\|_{\widehat{\gamma},i} \leq C\beta C_1^{1/\widehat{\gamma}} 4^i \|w^\beta\|_{1,i-1}, \quad \|w\|_{\beta\widehat{\gamma},i} \leq C^{\frac{1}{\beta}}\beta^{\frac{1}{\beta}} C_1^{1/\beta\widehat{\gamma}} 4^{\frac{i}{\beta}}\|w\|_{\beta,i-1}.$$

Replacing β by $\widehat{\gamma}^{i-1}\beta$, we derive

$$\begin{aligned}
\|w\|_{\beta\widehat{\gamma}^m,m} &\leq C^{\sum_{i=0}^{m-1}\frac{1}{\beta\widehat{\gamma}^i}} \cdot \prod_{i=0}^{m-1}(\beta\widehat{\gamma}^i)^{\frac{1}{\beta\widehat{\gamma}^i}} \cdot C_1^{\sum_{i=1}^{m}\frac{1}{\beta\widehat{\gamma}^i}} \cdot 4^{\sum_{i=0}^{m-1}\frac{i+1}{\beta\widehat{\gamma}^i}}\|w\|_{\beta,0}, \\
&= (C\beta)^{\frac{1}{\beta}\sum_{i=0}^{m-1}\frac{1}{\widehat{\gamma}^i}} \cdot C_1^{\frac{1}{\beta}\sum_{i=1}^{m}\frac{1}{\widehat{\gamma}^i}} \cdot \widehat{\gamma}^{\frac{1}{\beta}\sum_{i=0}^{m-1}\frac{i}{\widehat{\gamma}^i}} \cdot 4^{\frac{\widehat{\gamma}}{\beta}\sum_{i=0}^{m}\frac{i}{\widehat{\gamma}^i}}\|w\|_{\beta,0}.
\end{aligned}$$

From the elementary facts on power series we have

$$\sum_{i=0}^\infty \frac{1}{\widehat{\gamma}^i} = \frac{\widehat{\gamma}}{\widehat{\gamma}-1}, \quad \sum_{i=0}^\infty \frac{i}{\widehat{\gamma}^i} = \frac{\widehat{\gamma}}{(\widehat{\gamma}-1)^2},$$

consequently,

$$\begin{aligned}
\|w\|_{\infty,\infty} &\leq (C\beta)^{\frac{1}{\beta}\frac{\widehat{\gamma}}{\widehat{\gamma}-1}} \cdot C_1^{\frac{1}{\beta}\frac{1}{\widehat{\gamma}-1}} \cdot \widehat{\gamma}^{\frac{1}{\beta}\frac{\widehat{\gamma}}{(\widehat{\gamma}-1)^2}} \cdot 4^{\frac{\widehat{\gamma}}{\beta}\frac{\widehat{\gamma}}{(\widehat{\gamma}-1)^2}}\|w\|_{\beta,0}, \\
&= E_{n,k,T}(\beta) \cdot C_1^{\frac{1}{\beta}\frac{2}{\gamma-2}} \cdot \|w\|_{\beta,0}.
\end{aligned}$$

Since $I_\infty = [T/2, T]$ and $I_0 = [0, T]$, the corollary immediately follows. $\square$

Corollary 4.4. *Suppose that the integers n and k are greater than or equal to 2 and that $n+1 \geq k$. Consider the H^k mean curvature flow*

$$\frac{\partial}{\partial t}F(\cdot,t) = -H^k(\cdot,t)\nu(\cdot,t), \quad 0 \leq t \leq T \leq T_{\max} < \infty.$$

If

$$H(t) \geq \left(\frac{C_2}{k}\right)^{\frac{1}{k-1}} > 0, \quad \|kH^{k-1}(t)A^2(t)\|_{L^\infty(M\times[0,T])} < \infty,$$

along the H^k mean curvature flow for some uniform constant $C_2 > 0$, then there exists an uniform constant C_n, depending only on n, such that

$$\begin{aligned}
\|H(t)\|_{L^\infty\left(M\times\left[\frac{T}{2},T\right]\right)} &\leq E^{1/k}_{n,k,T}\left(\frac{n+k+1}{k}\right)\left(1+\|H(t)\|^{n+k+1}_{L^{n+k+1}(M\times[0,T])}\right)^{\frac{2}{\gamma-2}\frac{1}{n+k+1}} \\
&\cdot \quad \|H(t)\|_{L^{n+k+1}(M\times[0,T])}, \\
&\leq F_{n,k,T_{\max}} \cdot \|H(t)\|_{L^{n+k+1}(M\times[0,T])},
\end{aligned}$$

where

$$F_{n,k,T_{\max}} = E^{1/k}_{n,k,T_{\max}}\left(\frac{n+k+1}{k}\right)\left(1+\|H(t)\|^{n+k+1}_{L^{n+k+1}(M\times[0,T_{\max}))}\right)^{\frac{2}{\gamma-2}\frac{1}{n+k+1}}.$$

Proof. Let $f(x) = x^k : \mathbb{R}_+ \to \mathbb{R}$. From the evolution equation for $H(t)$,

$$\left(\frac{\partial}{\partial t} - \Delta_{f,t}\right)H(t) = |A(t)|^2_{g(t)} \cdot f(H(t)) + f''(H(t))|\nabla_t H(t)|^2_{g(t)},$$

we know that $G(t) = |A(t)|^2_{g(t)}$ and all conditions in Corollary 4.3 are satisfied. Hence there is an uniform constant C_n such that

$$\|H^k(t)\|_{L^\infty\left(M\times\left[\frac{T}{2},T\right]\right)} \leq E_{n,k,T}(\beta)C_1^{\frac{1}{\beta}\frac{2}{\gamma-2}}\|H^k(t)\|_{L^\beta(M\times[0,T])}$$

Taking kth root on both sides, we have

$$\|H(t)\|_{L^\infty\left(M\times\left[\frac{T}{2},T\right]\right)} \leq E^{1/k}_{n,k,T}(\beta)C_1^{\frac{2}{\gamma-2}\frac{1}{k\beta}}\|H(t)\|_{L^{k\beta}(M\times[0,T])}.$$

If we chose $\beta = \frac{n+k+1}{k} \geq 2$, then it follows that

$$\|H(t)\|_{L^\infty\left(M\times\left[\frac{T}{2},T\right]\right)} \leq E^{1/k}_{n,k,T}\left(\frac{n+k+1}{k}\right)\cdot C_1^{\frac{2}{\gamma-2}\frac{1}{n+k+1}}\|H(t)\|_{L^{n+k+1}(M\times[0,T])}.$$

By the definition of $E_{n,k,T}$ and C_1, the required inequality immediately follows. $\square$

Remark 4.5. *When $k = 1$, the assumption $n+1 \geq k$ is obvious. but for $k \geq 2$, this assumption is necessarily needed in our proof. In the forthcoming paper we may remove this condition.*

5. Proof of the main theorem and further remarks

The proof of our main theorem is similar to that given in [9], hence in this section we only give a sketch proof. From Hölder's inequality, it is sufficient to proof the theorem for $\alpha = n + k + 1$. Note that the quantity $\|H\|_{L^\alpha(M\times[0,T])}$ is invariant under the rescaling of the mean curvature flow

$$\widetilde{F}(p,t) = Q^{\frac{1}{k+1}} \cdot F\left(p, \frac{t}{Q}\right) \tag{5.1}$$

for $Q > 0$.

Suppose that the solution can not be extended over $T_{\max}$. Hence we know that $|A(t)|_{g(t)}$ is unbounded as $t \to T_{\max}$. Let $\lambda_i (i = 1, \cdots, n)$ denote the principle curvatures. Then

$$|A(t)|^2_{g(t)} = \sum_{i=1}^n \lambda_i^2 \leq \left(\sum_{i=1}^n \lambda_i\right)^2 = H^2(t).$$

Thus, $H^{k+1}(x,t)$ is also unbounded. We can chose a sequence of times $\{t^{(i)}\}_{i=1}^\infty$ with $\lim_{t\to\infty} t^{(i)} = T_{\max}$ and a sequence of points $\{x^{(i)}\}_{i=1}^\infty$ such that

$$Q^{(i)} = H^{k+1}(x^{(i)}, t^{(i)}) = \max_{(x,t)\in M\times[0,t^{(i)})} H^{k+1}(x,t) \to \infty.$$

Therefore there exists an integer i_0 such that $(Q^{(i)})^{\frac{2}{k+1}} t^{(i)} \geq 1$ for any $i \geq i_0$. Define

$$F^{(i)}(x,t) = (Q^{(i)})^{\frac{1}{k+1}} F\left(x, \frac{t-1}{(Q^{(i)})^{\frac{2}{k+1}}} + t^{(i)}\right), \quad i \geq i_0, \quad t \in [0,1].$$

Then a simple calculus shows that

$$\begin{aligned}
g^{(i)}(x,t) &= (Q^{(i)})^{\frac{2}{k+1}} g\left(x, \frac{t-1}{(Q^{(i)})^{\frac{2}{k+1}}} + t^{(i)}\right), \\
h^{(i)}_{pq}(x,t) &= (Q^{(i)})^{\frac{1}{k+1}} h_{pq}\left(x, \frac{t-1}{(Q^{(i)})^{\frac{2}{k+1}}} + t^{(i)}\right), \\
H^{(i)}(x,t) &= (Q^{(i)})^{-\frac{1}{k+1}} H\left(x, \frac{t-1}{(Q^{(i)})^{\frac{2}{k+1}}} + t^{(i)}\right),
\end{aligned}$$

where $g^{(i)}, h^{(i)}_{pq}$, and $H^{(i)}$ are the corresponding induced metric, second fundamental forms, and mean curvature, respectively. From the definition of $Q^{(i)}$ we must that

$$(H^{(i)}(x,t))^{k+1} \leq 1, \quad 0 \leq h^{(i)}_{pq}(x,t) \leq 1, \quad (x,t) \in M \times [0,1].$$

As in [9], we can find a subsequence of $\{M, g^{(i)}(t), F^{(i)}(t), x^{(i)}\}$, $t \in [0,1]$, converges to a Riemannian manifold $(\widetilde{M}, \widetilde{g}(t), \widetilde{F}(t), \widetilde{x})$, where $\widetilde{F}(t) : \widetilde{M} \to \mathbb{R}^{n+1}$ is an immersion. Since $(H^{(i)}(x,t))^{k+1} \leq 1$ on $M \times [0,1]$ for all $i \geq i_0$, it follows that $k(H^{(i)}(x,t))^{k-1}(A^{(i)}(x,t))^2$ is also bounded by 1 on $M \times [0,1]$ and any $i \geq i_0$. Consequently, we have, using Corollary 4.4,

$$\max_{(x,t)\in M^{(i)}\times\left[\frac{1}{2},1\right]} (H^{(i)}(x,t))^{k+1} \leq C\left(\int_0^1 \int_{M^{(i)}} |H^{(i)}(x,t)|^{n+k+1} d\mu_{g^{(i)}}(t)dt\right)^{\frac{k+1}{n+k+1}}$$

for some uniform constant C. Since the quantity $\|H\|^{n+k+1}_{L^{n+k+1}(M\times[0,T])}$ in invariant under the rescaling of the H^k mean curvature flow $Q^{\frac{1}{k+1}}F(\cdot,\frac{t}{Q})$, one has

$$\max_{(x,t)\in\widetilde{M}\times\left[\frac{1}{2},1\right]}\widetilde{H}^{k+1}(x,t)=\lim_{i\to\infty}\max_{(x,t)\in M^{(i)}\times\left[\frac{1}{2},1\right]}(H^{(i)}(x,t))^{k+1}\leq 0.$$

On the other hand, by our construction, we must have

$$\widetilde{H}^{k+1}(\widetilde{x},1)=\lim_{i\to\infty}(H^{(i)}(x^{(i)},1))^{k+1}=1.$$

This contradiction implies that the solution of the H^k mean curvature flow can be extended over $T_{\max}$.

Remark 5.1. *A natural question is to weaken the curvature condition on M. The main reason why we assume that the mean curvature of M has positive lower bound, comes from the term H^{k-1}; in the linear case $k=1$, this term must be a constant, but for the nonlinear case $k\geq 2$, we should impose some curvature conditions on M to guarantee the boundedness of such term.*

Our method mainly depends on [5]*, therefore, we may find other approaches to deal with the nonlinear case and to remove the positivity lower bound of the mean curvature on M. These will be treated with in the forthcoming paper* [7].

References

1. A. A., Cooper, *Mean curvature blow up in mean curvature flow*, preprint, arXiv: math.DG/0902.4282.
2. L. Grafakos, *Classical Fourier Analysis*, 2nd ed., Graduate Texts in Mathematics, Springer, 2008.
3. G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Diffeential Geom. **20** (1984), no.1, 237–266.
4. P. Lax, *Formuation and propagation of shock waves, the role of entropy: A history of shock waves and its future development*, First Ahlfors lecture (Public), organized by Shing-Tung Yau and Horng-Tzer Yau, Science Center Lecture Hall D, Harvard, 4:30 pm–5:30 pm, Oct.5., 2009.
5. N. Q. Le, N. Sesum, *On the extension of the mean curvature flow*, preprint, arXiv: math.DG/0905.0936v2.
6. G. Michael, L. Simon, *Sobolev and mean-value inequalities on generalized submanifolds of* $\mathbb{R}^n$, Comm. Pure Appl. Math. **26**(1973), 361-379.
7. A. Ostergaard, Y. Li, *On an extension of the H^k mean curvature flow II*, in preparation.
8. K, Smoczyk, *Harnack inequalities for curvature flows depending on mean curvature*, New York J. Math. **3** (1997), 103–118.
9. H. W. Xu, F. Ye, E. T. Zhao, *Extend mean curvature flow with finite integral curvature*, preprint, arXiv: math.DG/0905.1167v1.

Department of Mathematics, Harvard University, Cambridge, MA 70803
E-mail address: yili@math.harvard.edu